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MHD waves on the Sun

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Abstract : Theoretical aspects of wave propagation in an atmosphere with a vertical magnetic field are reviewed. Starting from the ideal magnetohydrodynamic equations, the differential equation governing linear oscillations is presented for a stratified medium with a vertical magnetic field. In order to obtain physical insight into the nature of the different types of waves, it is instructive to first consider certain simple cases, for which the wave equation can be solved analytically. The dispersion relations for these modes are examined. Next, the case of an isothermal stratified atmosphere is considered in the limit of strong and weak magnetic fields. The effect of a magnetic field on the modal structure is highlighted. Coupling of different types of modes is examined. Solutions for the general case corresponding to the atmosphere in the umbra of a sunspot are also presented and related to observations of umbral oscillations. Finally, wave propagation in intense flux tubes is briefly discussed.

Keywords : Solar magnetic fields, oscillations, magnetohydrodynamics

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1. Introduction

The study of oscillations on the Sun has witnessed impressive progress in recent years. With the development of sophisticated observational techniques, it is now possible to detect waves in the solar atmosphere over a wide range of frequencies and length scales. The discovery of 5 min global oscillations and their theoretical interpretation has generated considerable interest in wave phenomena and led to the growth of a whole branch of knowledge, known as helioseismology, devoted exclusively to this aspect. The various modes, which are localised in different regions of the Sun, provide valuable information about physical conditions in the solar interior. However, in this review, we shall be concerned mainly with the surface layers *i.e.*, the photosphere, which is most accessible to direct observations (in the optical wavelength band). It is also well known that the photosphere is threaded with strong magnetic fields. The aim of this paper is to review the nature of oscillations associated with these magnetized regions.

Historically, the study of wave motions on the Sun dates back to the 1940's, when theorists grappled with the question of understanding why the corona is much hotter than the underlying photosphere. Typically, the temperature of the corona is some million degrees, whereas that of the photosphere is only a few thousand degrees. The earliest theories were based upon the hypothesis that the convection zone is a rich source of sound waves. As these waves travel upwards, their amplitudes increase (because of energy conservation) owing to the fall in density and eventually they form shocks and dissipate their energy very rapidly. For a variety of reasons, which will not be discussed here, pure acoustic wave theories are dead and most modern theories of coronal heating involve the magnetic field in some way or the other and depend upon magnetic waves to carry the energy flux.

It is now generally accepted, on the basis of observations (for a recent review see [1] and references therein) that the magnetic field on the solar surface occurs in the form of discrete elements known as flux tubes. Diameters of these tubes range from several thousand kilometers, corresponding to sunspots, down to a few hundred kilometers, associated with intense flux tubes or fibrils. There are also tubes with intermediate sizes, such as pores and faculae. Despite the fact that these elements have different horizontal dimensions, one feature that is common to all of them is that the magnetic field in the tubes is fairly strong, with field strengths in the kilogauss range. Flux tubes on the sun exhibit a rich variety of oscillations, of which umbral oscillations with periods in the 2-3 min band and penumbral waves are perhaps the most well known. In addition, oscillations have also been observed in intense flux tubes, though not as extensively as in sunspots, because intense tubes are hard to resolve from ground based instruments. In the ensuing sections, we shall examine some of the properties of such oscillations. However, before doing this, it might be worthwhile to briefly mention some of the reasons for studying magnetic waves on the Sun. They are :

- Waves are likely candidates for transporting energy from the sub-surface layers to the upper regions of the atmosphere.
- Waves provide a powerful diagnostic for probing physical conditions within flux tubes, particularly in the deeper layers, normally inaccessible to observations.
- Lastly, the Sun can be regarded as an astrophysical observatory, which can yield important clues to understanding wave phenomena in laboratory magnetised plasma, in addition to being a site for testing new theories.

Qualitatively, the waves that we are dealing with are called magneto-acoustic-gravity (MAG for short), reflecting the three distinct forces that are present, *viz.*, magnetic, pressure and gravity. Without gravity, the theoretical analysis is straightforward. However, when stratification due to gravity is taken into account, the mathematical problem becomes considerably more complicated. Historically, the normal modes of a stratified medium with a vertical magnetic field began with the analysis of Ferraro and Plumpton [2]. Since then there have been several papers and reviews on this subject [3-18]. Despite the considerable attention that this topic has received, it would still be premature to say that we have a complete understanding of MAG waves. In the following sections, we shall quantitatively discuss some of the fundamental aspects related to MAG waves. The treatment adopted in this review is by no means exhaustive and reflects to some extent the author's own interests.

The plan of this review is as follows : in Section 2, the relevant equations governing oscillations will be presented. Analytic solutions corresponding to special cases are examined. In Sections 3 and 4, waves in a medium with zero gravity (unstratified) and in a stratified atmosphere with no magnetic field are treated respectively. Section 5 examines oscillations in an isothermal stratified atmosphere with a vertical magnetic field. The analysis clearly highlights the effect of a magnetic field on the modal structure of a non-magnetic atmosphere. The most general case is treated in Section 6, where numerical results for umbral oscillations are presented. Finally, oscillations in intense flux tubes are discussed briefly in Section 7.

2. Equations

Let us consider a stratified plasma with a uniform vertical magnetic field. Using a fluid description and assuming an ideal plasma (*i.e.*, inviscid and with infinite conductivity), the relevant MHD equations are

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \boldsymbol{v}) = 0, \tag{1}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{g} - \nabla p + \frac{1}{4\pi} \nabla \times \boldsymbol{B} \times \boldsymbol{B}, \tag{2}$$

$$\frac{d}{dt} \left(\frac{p}{\rho^{\gamma}} \right) = 0, \tag{3}$$

$$\nabla . \boldsymbol{B} = 0, \tag{4}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t}, \tag{5}$$

$$\boldsymbol{E} + \frac{1}{c} (\boldsymbol{v} \times \boldsymbol{B}) = 0, \tag{6}$$

where ρ is the mass density of the fluid, $\boldsymbol{v}$ is the velocity, p is the pressure, $\boldsymbol{B}$ is the magnetic field strength and $\boldsymbol{E}$ is the electric field. The constants g, c and γ refer to acceleration due to gravity, the speed of light and the ratio of specific heats respectively. Eq. (1) express continuity of mass, eq. (2) is the equation of motion, eq. (3) corresponds to the assumption of an adiabatic fluid, eq. (4)–(5) are Maxwell's equations and eq. (6) is Ohm's law for an infinitely conducting plasma.

2.1. *Equilibrium :*

At $t = 0$ (*i.e.*, the unperturbed state) let us assume that the fluid is in hydrostatic equilibrium and that all physical quantities have only a z dependence, apart from the magnetic field, which we take to be constant. The equilibrium state can be determined by solving eq. (1)–(6), without the time derivatives. Let us also assume that the velocity is zero in the unperturbed atmosphere. Thus, in equilibrium we have

$$\frac{dp}{dz} = -\rho g \tag{7}$$

assuming that gravity acts in the z direction, which is chosen to point away from the Sun. For simplicity, let us assume that the equilibrium atmosphere is isothermal *i.e.*, the temperature is constant with height. This implies that the sound speed defined as $c_s = \sqrt{\gamma p/\rho}$ is constant with z and that ρ has the following height dependence

$$\rho = \rho_0 \exp^{(-z/H)}, \tag{8}$$

where the subscript zero refers to values at $z = 0$ and $H = p/\rho g$ is defined as the scale height of the atmosphere, which is constant for an isothermal medium.

2.2. Wave equation :

We now consider wave propagation in an ideal stratified plasma. This is accomplished by considering small perturbations about the equilibrium. Let $\vec{\xi}$ denote a Lagrangian displacement of a fluid element from its equilibrium position. This displacement is related to the velocity through the equation

$$v = \frac{\partial \vec{\xi}}{\partial t}. \tag{9}$$

Expressing all physical variables in eq. (1)–(6) as the sum of an unperturbed part and a small perturbation and retaining only first order terms in the perturbations, we arrive at the following differential equation

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = -F\left(\vec{\xi}\right), \tag{10}$$

where

$$F(\vec{\xi}) = \nabla \delta p - g \delta\rho - \frac{1}{4\pi} (\nabla \times \delta B) \times B, \tag{11}$$

$$\delta\rho = -\rho \nabla . \vec{\xi} - \vec{\xi}\, \nabla \rho, \tag{12}$$

$$\delta p = -\gamma p \nabla . \vec{\xi} - \vec{\xi} . \nabla p, \tag{13}$$

$$\delta B = \nabla \times (\vec{\xi} \times B), \tag{14}$$

where $\delta\rho$, δp, δB denote Eulerian perturbations in density, pressure and magnetic field respectively.

In cartesian geometry, the linearised equations for MAG waves, assuming that the displacements have the form $\exp[i(\omega t - kx)]$, can be written as [2]

$$\left[\vartheta_A^2 \frac{d^2}{dz^2} - \left(c_S^2 + \vartheta_A^2\right)k^2 + \omega^2\right]\xi_x - ik\left(c_S^2 \frac{d}{dz} - g\right)\xi_z = 0, \tag{15}$$

$$\left[c_S^2\frac{d^2}{dz^2}-\gamma g\frac{d}{dz}+\omega^2\right]\xi_z-ik\left[c_S^2\frac{d}{dz}-(\gamma-1)g\right]\xi_x=0, \tag{16}$$

$$\left(\omega^2+\vartheta_A^2\frac{d^2}{dz^2}\right)\xi_y=0, \tag{17}$$

where k is the horizontal wave number, ω is the frequency, and ϑ_A is the Alfvén speed defined as $\vartheta_A = B/\sqrt{4\pi\rho}$. Eqs. (15) and (16) govern the propagation of MAG waves in a stratified atmosphere.

Eq. (17), which is decoupled from the other equations, describes purely transverse waves known as Alfvén waves, bearing the name of the person who first discussed their theoretical properties. We have implicitly assumed that the propagation and motions of the MAG modes are confined to the x–z plane and of the Alfvén modes to the y–z plane. This involves no loss of generality.

It is instructive to examine the solution of the wave equations (15)–(17) for certain special cases, which can be treated analytically.

3. Zero gravity (unstratified medium)

It is well known that in a unstratified magnetized plasma, there are basically only three types of waves, *viz.*, fast, slow and Alfvén . The dispersion relation can be obtained fairly trivially from eqs. (15)–(17) and is

$$\left(\omega^2-k_z^2v_A^2\right)\left[\omega^4-\omega^2\left(c_S^2+v_A^2\right)k_t^2+k_z^2k^2c_S^2v_A^2\right]=0, \tag{18}$$

where k_z is the vertical wave number and k_t is the total wave number defined as $k_t = \sqrt{k^2 + k_z^2}$. Equating the first term in eq. (18) to zero yields the dispersion relation for Alfvén waves. These waves as discussed earlier are decoupled from the other modes.

On equating the second term in eq. (18) to zero, the dispersion relation for fast and slow modes is obtained. It is instructive to look at this relation in the *thin flux tube limit i.e.*, when $k_z/k \rightarrow 0$. In this limit ω_f and ω_s, the fast and slow mode frequencies respectively, are

$$\omega_f^2=\frac{k_t^2\left(c_S^2+v_A^2\right)}{2}, \tag{19a}$$

$$\omega_s^2=k_z^2c_T^2, \tag{19b}$$

where $c_T = c_S v_A \,/\sqrt{c_S^2+v_A^2}$ is known as the tube speed.

4. Unmagnetized medium (B = 0)

4.1. *Isothermal atmosphere :*

The dispersion relation for a stratified isothermal medium is

$$\omega^4 - \left(\omega_a^2 + k_t^2 c_S^2\right)\omega^2 + k_t^2 c_S^2 \omega_{BV}^2 \sin^2\theta = 0, \tag{20}$$

where

$$\omega_a = \frac{\gamma g}{2c_S}, \tag{21}$$

$$\omega_{BV} = \frac{(\gamma - 1)^{1/2} g}{c_S}, \tag{22}$$

and θ is the angle between the vertical (z–axis) and $\boldsymbol{k}$. The quantities ω_a and ω_{BV} are referred to as the acoustic cut-off and Brunt-Väisälä frequencies respectively. Eq. (20) is the dispersion relation for atmospheric waves. It is useful to write eq. (20) as

$$k_z^2 = \frac{\left(\omega^2 - \omega_a^2\right)}{c_S^2} + k^2\left(\frac{\omega_{BV}^2}{\omega^2} - 1\right). \tag{23}$$

Wave propagation occurs only in those regions where $k_z^2 > 0$. When $kH >> 1$, the dispersion relation factorizes out into two expressions

$$\omega^2 = k_t^2 c_S^2, \tag{24}$$

$$\omega^2 = \omega_{BV}^2 \sin^2\theta, \tag{25}$$

where eqs. (24) and (25) correspond to acoustic (p) and gravity (g) modes respectively. In the opposite limit when $kH << 1$ the following relations are obtained

$$\omega^2 = \omega_a^2\ , \tag{26}$$

$$\omega^2 = \frac{\omega_{BV}^2}{\omega_a^2} c_S^2 k^2 . \tag{27}$$

4.2. Polytropic atmosphere :

We now consider wave propagation in a polytopic atmosphere, without a magnetic field. In such an atmosphere, the pressure and density are related according as follows

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{1+1/l} \tag{28}$$

where l is the polytropic index and the subscript zero refers to values at $z = 0$. This case was first investigated by Lamb [19], who found an analytic solution to the problem in terms of confluent hyper geometric functions. For large horizontal wave numbers the dispersion relation can be factorized once again into p and g modes. In the limit of high vertical mode order n, it is [20] *p-modes*

$$\omega^2 = 2\gamma g k \frac{\left(n+\frac{l}{2}\right)}{(l+1)}, \tag{29}$$

g-modes

$$\omega^2 = \frac{gk}{2} \frac{\left(l-\frac{l+1}{\gamma}\right)}{\left(n+\frac{l}{2}\right)}. \tag{30}$$

where l is the polytropic index of the fluid. It is easy to show from eq. (30) that for an adiabatically stratified fluid ($l = 3/2$), the g-modes are neutral.

5. Magnetized medium with isothermal stratification

The problem of wave propagation in an isothermal stratified atmosphere with a vertical magnetic field was first investigated by Ferraro and Plumpton [2], who obtained analytic expressions in the asymptotic limits of strong and weak magnetic fields. For the general case of arbitrary field strengths, an analytic solution in terms of Meijer function was found by Žugžda [5]. In order to obtain a physical insight into this problem, let us first consider the solutions in the asymptotic limits of weak and strong fields. Since the differential equation is of fourth order, there are in general four linearly independent solutions.

5.1. Strong field limit :

For a strong field, such that $c_S/v_A \to 0$, the solutions for the vertical component of the wave amplitudes are of the form

$$\xi_z^{1,2} \sim \exp\left[-(1 \pm i\alpha)\frac{z}{2H}\right], \tag{31}$$

$$\xi_z^{3,4} \sim e^{\pm kz}, \tag{32}$$

where $\Omega = \omega H/c_s$ and $\alpha = \sqrt{4\Omega^2 - 1}$. Solutions given by eq. (31) are essentially slow waves, modified by gravity, whereas those given by eq. (32) correspond to gravity modified fast modes. The latter are evanescent in the vertical direction (purely growing solutions are unphysical and can be discarded). It is interesting to note that the atmosphere has a cutoff frequency, which is the same as that for a atmospheric wave.

5.2. Weak field limit :

For a weak field, such that $v_A/c_S \cdot$ 0, the solutions for the vertical component of the wave amplitudes are of the form

$$\xi_z^{1,2} \sim \theta^{(-1\pm 2iK_z)}, \tag{33}$$

$$\xi_z^{3,4} \sim \mp \frac{e^{\pm 2i\theta}}{\theta^{3/2}}, \tag{34}$$

where

$$K_z^2 = \Omega^2 - K^2\left(1 - \frac{\Omega_{BV}^2}{\Omega^2}\right) - \frac{1}{4}, \tag{35}$$

$\theta = \Omega c_s / v_A$, $K = kH$ and $\Omega_{BV} = (\gamma - 1)/\gamma^2$ is the Brunt-Väisälä frequency (in dimensionless units). The case $\Omega = \Omega_{BV}$ requires special treatment, which can be found in [2].

Let us first consider the solutions corresponding to eq. (33). These solutions, representing propagating waves for $K_z^2 > 0$, *i.e.*, when $\Omega > \Omega_a$ or $\Omega < \Omega_{BV}$, can be recognized as the usual p and g modes in an unmagnetized isothermal plasma. For $\Omega_{BV} < \Omega < \Omega_a$, the solutions represent purely evanescent waves.

Turning our attention to the solutions given by eq. (34), we find that these modes arise solely due to the presence of the magnetic field. They are approximately transverse. Physically, these modes can be interpreted as gravity modified slow modes in a weak magnetic field.

Normal modes :

We examine the normal modes of a stratified atmosphere in a weak magnetic field, in order to delineate the influence of the field on the oscillation spectrum. For simplicity, let us assume rigid boundary conditions such that

$$\xi_x = \xi_z = 0 \text{ at } z = 0 \text{ and } z = D, \tag{36}$$

where D is the height of the top boundary in units of H.

Using eqs. (36) as boundary conditions, it can be shown that the dispersion relation is [21]

$$\begin{aligned}
&(\Omega^2 - K^2)\sin\tilde{\theta}\sin(K_z D) \\
&= \frac{\varepsilon}{\Omega} e^{D/4}\Big\{2K_z K^2\left[\cosh(D/4)\cos\tilde{\theta}\cos(K_z D) - 1\right] \\
&+ 2\sinh(D/4)\cos\tilde{\theta}\sin(K_z D)\left[M(\Omega^2 - K^2) - K^2\left(\frac{1}{\gamma} - \frac{1}{2}\right)\right]\Big\} \\
&+ O\left(\frac{\varepsilon^2}{\Omega^2}\right) = 0 ,
\end{aligned} \tag{37}$$

where

$$\varepsilon = \frac{v_{A,0}}{c_S} \qquad r = \left| 1 \quad \frac{\Omega_{BV}^2}{\Omega^2} \right.$$

$$\theta_0 = \theta(0), \qquad \theta_D = \theta(D), \qquad \tilde{\theta} = 2(\theta_0 - \theta_D),$$

$$M = K^2 \frac{\Omega_{BV}^2}{\Omega^2} - \frac{1}{16}.$$

For $\varepsilon << \Omega$, the dispersion relation to lowest order in ε/Ω becomes

$$(\Omega^2 - K^2)\sin\,\tilde{\theta}\,\sin(K_z D) = 0, \tag{38}$$

Eq. (38) admits the following solutions

$$\sin(K_z D) = 0, \tag{39}$$

$$\Omega = K, \tag{40}$$

$$\sin\,\tilde{\theta} = 0, \tag{41}$$

Let us first consider the solution given by eq. (37) which implies that $K_z D = n\pi$, where n is an integer and denotes the order of the mode. Use of eq. (35) yields the usual relation for p- and g-modes, which is

$$\Omega_i^4 - \Omega_i^2\left(K_t^2 + \frac{1}{4}\right) + K^2\Omega_{BV}^2 = 0 \quad (i = p, g), \tag{42}$$

where $K_t^2 = K_z^2 + K^2$.

The solution corresponding to eq. (40) can easily be recongnised as the Lamb mode in an unmagnetized atmosphere. Let us denote this frequency as Ω_L. The Lamb mode in a non-magnetic atmosphere propagates horizontally and is evanescent in the vertical direction. It does not satisfy the boundary conditions (36). However, this can be achieved by adding a small contribution from the magnetic modes.

Turning our attention to the solutions given by eq. (41), we find that these modes arise solely due to the presence of the magnetic field. On applying the boundary conditions given by eqs. (36), we find that the magnetic modes have frequencies

$$\Omega_m = \frac{\varepsilon\, n\pi}{2s} \tag{43}$$

where $s = (1 - \varepsilon^{-D/2})$. These modes are approximately transverse, since it can be shown [21] that

$$\frac{\xi_x^{1,2}}{\xi_z^{1,2}} \sim O(\theta)$$

Physically, these modes can be interpreted as gravity modified slow modes in a weak magnetic field.

We now consider the corrections to the lowest order frequencies obtained from eq. (38).

Frequency correction to the p- and g- modes :

Let us write

$$\Omega = \Omega_i + \delta\Omega_i \tag{44}$$

where Ω_i ($i = p,g$) is the solution of eq. (39), and $\delta\Omega_i$ denotes the first order correction to Ω due to the magnetic field. Substituting eq. (44) in the dispersion relation (eq. 37), yields an expression for $\delta\Omega_i$, which is

$$\delta\Omega_i = \frac{2\varepsilon e^{D/4} K_z^2 K^2}{D\Omega_i^2(\Omega_i^2 - K^2)}\left(1 - \frac{K^2\Omega_{BV}^2}{\Omega_i^4}\right)^{-1}\left[(-1)^{n+1}\operatorname{cosec}\tilde{\theta} + \cosh(D/4)\cot\tilde{\theta}\right], \tag{45}$$

where $K_z = n\pi/D$.

In the limit $K \to 0$, $\Omega_g \to 0$ and $\Omega_p \to n\pi/D$ (for $n\pi/D >> 1$). The correction to Ω_p in this limit is

$$\delta\Omega_p = \frac{2\varepsilon D e^{D/4} K^2}{n^2\pi^2}\left[(-1)^{n+1}\operatorname{cosec}\tilde{\theta} + \cosh(D/4)\cot\tilde{\theta}\right], \tag{46}$$

The frequency correction, for small K varies as K^2 and n^{-2}, becoming negligibly small for high orders. In the limit $K \to 0$, eq. (45) cannot be used for the g-modes since these have very small frequencies ($\Omega_g \sim K$).

Let us now consider the limit of large K, so that $K >> \Omega$ and $K >> K_z$. In this limit, $\Omega_g \to \Omega_{BV}$ and

$$\delta\Omega_g = \frac{2\varepsilon e^{D/4} n^2\pi^2}{D^3 K^2}\left[(-1)^{n+1}\operatorname{cosec}\tilde{\theta} + \cosh(D/4)\cot\tilde{\theta}\right], \tag{47}$$

Eq. (47) hold for K in the range $\Omega << K << \Omega_{BV}^{3/2}\ \varepsilon^{-1/2}$

The frequency correction, for large K varies as K^{-2} and n^2, and, therefore, increases with mode order. The correction to Ω_p cannot be obtained from eq. (45) since $\Omega_p \sim K$

Frequency correction to the Lamb mode :

Let us now consider the frequency correction to the Lamb mode with frequency $\Omega_L = K$. The frequency correction to this mode is

$$\delta\Omega_L = \frac{\varepsilon}{D} e^{D/4}\tilde{K}_z\Big[\cosh(D/4) D\cot\tilde{\theta}\coth(\tilde{K}_z)$$

$$-\operatorname{cosec}\tilde{\theta}\operatorname{cosech}(\tilde{K}_z) - \sinh(D/4)\Big] + O\left(\frac{\varepsilon}{K}\right), \tag{48}$$

where $\tilde{K}_z = (1/\gamma - 1/2)$. Eq. (48) is valid for $\varepsilon^{-1} >> K >> \varepsilon$. For $K < \varepsilon$, the condition $\delta\Omega_L/\Omega_L << 1$ does not hold, whereas for $K >> \varepsilon^{-1}$, the second order term on the right hand side of eq. (48) dominates over the first order term.

Frequency correction to the magnetic mode :

The frequency correction to Ω_m is

$$\delta\Omega_m = \frac{\varepsilon^2 e^{D/4}}{2s\Omega_m(\Omega_m^2 - K^2)}\Big\{2K_z K^2\big[(-1)^{n+1}\mathrm{cosec}(K_z D) + \cosh(D/4)\cot(K_z D)\big]$$

$$+2\sinh(D/4)\left[M\left(\Omega_m^2 - K^2\right) - K^2\left(\frac{1}{\gamma} - \frac{1}{2}\right)\right]\Big\}, \qquad (49)$$

where Ω_m is given by eq. (43).

In the limit $K << \Omega_m$, we find

$$\delta\Omega_m = \frac{\varepsilon e^{D/4}}{n\pi}\Big\{-\frac{\sinh(D/4)}{8} + \frac{K^2}{\Omega_m^2}\Big[2K_z\big((-1)^{n+1}\mathrm{cosec}(K_z D)$$

$$+\cosh(D/4)\cot(K_z D)\big) + 2\sinh(D/4)\left(\frac{1}{2} - \frac{1}{\gamma^2}\right)\Big\} \qquad (50)$$

Table 1. Values of c_S, v_A and $\theta = \omega H/v_a$ at different z [13].

z (km)	c_S (km s^{-1})	v_A (km s^{-1})	θ
−500 ...	10.42	2.81 (4.21)	3.84 (2.56)
−300 ...	9.61	3.63 (5.45)	2.44 (1.62)
−100 ...	8.24	4.55 (5.45)	1.16 (0.77)
0 ...	6.52	5.84 (8.22)	0.59 (0.40)
100 ...	5.87	9.13 (13.70)	0.29 (0.19)
300 ...	5.94	34.82 (52.23)	0.08 (0.05)

Note : H is the scale height and ω is the frequency, assuming B = 2000 G and a wave period of 180 s. The numbers in parentheses correspond to B = 3000 G

where $K_z = \sqrt{\Omega_m^2 - 1/4}$.

In the opposite limit, when $K >> \Omega_m$, we have

$$\delta\Omega_m = \frac{\varepsilon^2 K^2}{\Omega_m^3}\frac{\sinh(D/4)\,\Omega_{BV}^2 e^{D/4}}{s}, \qquad (51)$$

The frequency correction given by eq. (51) holds as long as $\delta\Omega_m/\Omega_m << 1$. This restricts the validity of eq. (51) to K in the range $\Omega_m << K << \Omega_m^2\ \varepsilon^{-1}$ or roughly when $n\pi\varepsilon << K << n^2\pi^2\varepsilon$.

6. Magnetized atmosphere with general stratification

We now consider waves in a stratified and magnetized atmosphere of the most general kind. The nature of the waves in this case is much more difficult to understand and cannot be analysed in terms of the elementary modes discussed in the previous section. In an atmosphere, where the ratio of the sound to Alfvén speed changes with height, a continuous transformation of the wave modes occurs. This mode transformation is particularly effective in layers where $\omega H/v_A \sim 1$ [22]. Table 1 (taken from [13] hereafter Paper II) gives the values of v_A, c_S and θ, where $\theta = \omega H/v_A$, at various z for B = 2000 G and ω = 3.49 rad s^{-1} (corresponding to a period of 180 s). The numbers in parentheses give the values for B = 3000 G. We find that the "transformation" region occurs deeper in the umbra for B = 3000 G.

Thus, when attributing a fast or slow character to a mode, it should be born in mind that this is local property, and applicable only in layers separated from the "transformation region". Incidentally, decreasing the frequency, moves this region lower down into the atmosphere.

Let us consider the solution of the wave equation in the most general case. For the sake of illustration, we choose an equilibrium atmosphere corresponding to the umbra of a typical sunspot. The normal modes of this atmosphere can be obtained by numerically solving eqs. (15)-(16). Details of the method can be found in Paper II. Figure 1 (taken from Paper II) shows the diagnostic diagram *i.e.*, the variation of the wave frequency ν (where $\nu = \omega/2\pi$) with k for umbral oscillations, for B = 2000 G (solid lines) and B = 3000 G (dashed lines), assuming rigid boundaries.

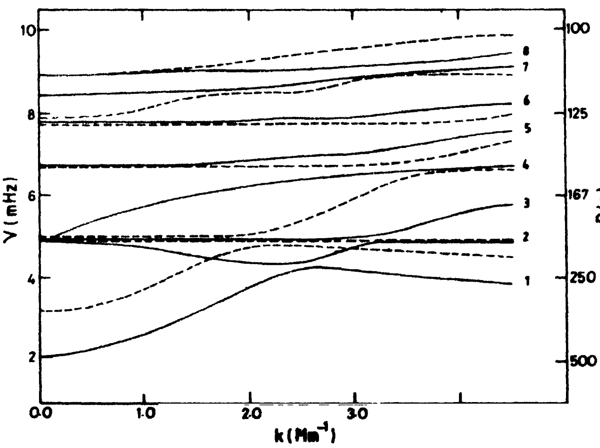

Figure 1. Variation of υ with k in the umbra of typical sunspot for B=2000 G (solid lines) and B=3000 G (dashed lines), corresponding to υ_{A0}/c_{S0}=0.84 and υ_{A0}/c_{S0}=1.26 respectively, where the subscript zero corresponds to z=0. The right-hand scale denotes the periods and the numbers besides the curves correspond to the order n of the solution, with respect to B=2000 G (from [13])

The various numbers besides the solid curves denote the orders (n) of the solutions (with respect to B = 2000 G). An interesting aspect of the solutions in that curves of different orders do not intersect *i.e.*, accidental degeneracy does not occurs. In fact, when curves of adjacent orders approach one another, an *avoided crossing* occurs. Physically, this phenomenon can be understood in terms of coupling between modes confined to different regions of the atmosphere [2,3]. The existence of *avoided crossings* in the context of magnetoatmospheric waves appears to have been only recently notices [12]. A detailed analysis of the nature of the waves in the vicinity of *avoided crossings* has been given in [21].

Certain properties of the solutions can be discerned by an inspection of Figure 1. For instance there are regions in the diagnostic diagram where the frequency hardly varies with k. These portions roughly correspond to modes which are dominantly of the slow type. In the limit of large k, the dispersion relation for the slow mode is given by eq. (19b). When $k \to \infty$, $\omega \to \infty$ for the fast mode, so that the two sets of modes are well separated in frequency. Thus, for large horizontal wave number, we expect the low order modes to be of the slow type, whose frequencies depend on B through the tube speed c_T. The magnetic nature of a mode can be seen by looking, in this limit, at the shift in frequency, when the magnetic field is varied. We find that the frequency of the n=2 mode is unaffected by a change in the field, suggesting that this mode is dominantly acoustic in character. For the other modes, however, the frequency increases with B, although the dependence on field strength is fairly weak.

7. Oscillations in intense flux tubes

In the previous sections, we considered wave propagation in a stratified atmosphere with a uniform vertical magnetic field. Let us now consider the case of a single vertical isolated flux tube surrounded by an unmagnetized medium. Owing to the stratification, the tube will flare out with height. This problem in its generality is at present intractable. However, some progress can be made if one resorts to the *thin flux tube approximation*, which essentially assumes constance of all quantities in the horizontal direction. In this approximation, the ideal MHD equations for axisymmetric longitudinal disturbances are [24,25]

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B}\right)+\frac{\partial}{\partial z}\left(\frac{\rho v}{B}\right)=0, \tag{52}$$

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial z}-\rho g, \tag{53}$$

$$\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial z}-c_S^2\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial z}\right)=0, \tag{54}$$

$$p+\frac{B^2}{8\pi}=p_e, \tag{55}$$

$$BA=\text{constant}, \tag{56}$$

where v and B denote the vertical components of the velocity and magnetic field respectively in the tube with cross-sectional area A and p_e denotes the external pressure.

For linear motions, eqs. (52)–(56) can be manipulated to yield the following equation [26]

$$\frac{\partial^2 Q}{\partial t^2} - c_T^2 \frac{\partial^2 Q}{\partial z^2} + \omega_L^2 Q = 0, \tag{57}$$

where

$$Q = \left(\frac{\rho c_T^2}{B}\right)^{1/2} v, \tag{58}$$

and

$$\omega_L^2 = \frac{c_T^2}{H^2}\left[\frac{c_S^2}{v_A^2}\left(\frac{\gamma - 1}{\gamma^2}\right) + \frac{9}{16} - \frac{1}{2\gamma}\right]. \tag{59}$$

For an isothermal medium, c_T and ω_L are constants. It is then easy to show that eq. (57), y: 'ds the dispersion relation

$$\omega^2 = \omega_L^2 + c_T^2 k_z^2, \tag{60}$$

where k_z denotes the vertical wave number. From eq. (60), we find that ω_L corresponds to the cutoff frequency for tube modes. For an unstratified medium, $\omega_L = 0$ and we recover the slow mode dispersion relation (*cf.* eq. [19b]). The cutoff frequency for longitudinal disturbances in a thin flux tube can be expressed to terms of $\omega_a = c_S/2H$, the usual cutoff frequency in a field-free atmosphere, as

$$\omega_L^2 = 4\omega_a^2 \frac{c_T^2}{c_S^2}\left[\frac{c_S^2}{v_A^2}\left(\frac{\gamma - 1}{\gamma^2}\right) + \frac{9}{16} - \frac{1}{2\gamma}\right]. \tag{61}$$

For typical parameters in the photosphere, it turns out that $\omega_L \sim \omega_a$.

So far we have been dealing with the axisymmetric mode, also referred to as the *sausage moee*. Let us now consider the kink modes of an intense flux tube, These modes are transverse and for an isothermal medium are governed by the following differential equation [27]

$$(2\beta + 1)\frac{\partial^2 v_\perp}{\partial t^2} = -g\frac{\partial v_\perp}{\partial z} + 2gH\frac{\partial^2 v_\perp}{\partial z^2}, \tag{62}$$

where $v_\perp$ denotes the transverse velocity amplitude and $\beta = 8\pi p/B^2$. Assuming a variation of the form

$$v_\perp \sim \exp(i\omega t - ik_z z + z/4H), \tag{63}$$

we obtain the following dispersion relation from eq. (62)

$$\omega^2 = \omega_K^2 + c_K^2 k_z^2, \tag{64}$$

where

$$c_K^2 = \frac{v_A^2}{2\beta + 1}, \tag{65}$$

$$\omega_K^2 = \frac{g}{8H(2\beta+1)}. \tag{66}$$

From eq. (66), we find that analogous to longitudinal oscillations, kink oscillations travel with the a speed c_K and have a cut off frequency ω_K. Eq. (66) can also be written as

$$\omega_K^2 = \left[\frac{1}{2\gamma(2\beta+1)}\right]\omega_a^2, \tag{67}$$

In general, $\omega_K < \omega_a$, which implies that lower frequencies can propagate up to greater heights in flux tubes than in the field free medium. This can have important implications for coronal heating, since there is considerable power in photospheric oscillations at periods below the acoustic cut-off period.

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